

WEAKENING THE TOPOLOGY OF A LIE GROUP

BY

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ABSTRACT. With any topological group $(G, \mathcal{U})$ one can associate a locally arcwise-connected group $(G, \mathcal{U}^*)$, where $\mathcal{U}^*$ is stronger than $\mathcal{U}$. $(G, \mathcal{U})$ is a *weakened Lie (WL) group* if $(G, \mathcal{U}^*)$ is a Lie group. In this paper the author shows that the WL groups with which a given connected Lie group $(L, \mathcal{T})$ is associated are completely determined by a certain abelian subgroup H of L which is called *decisive*. If L has closed adjoint image, then H is the center $Z(L)$ of L ; otherwise, H is the product of a vector group V and a group J that contains $Z(L)$. $J/Z(L)$ is finite (trivial if L is solvable). We also discuss the connection between these theorems and recent results of Goto.

1. Introduction. Gleason and Palais [1] have shown how to associate with any topological group $(G, \mathcal{U})$ a locally arcwise-connected group $(G, \mathcal{U}^*)$ and proved that the group thus associated with a finite-dimensional metric group must be Lie. In this paper we describe the groups for which the associated locally arcwise-connected group is a connected Lie group. As we shall see in §3, this problem is equivalent to the following question: Given a connected Lie group $(L, \mathcal{T})$, in what ways can $\mathcal{T}$ be weakened and remain Hausdorff? Our principal result is that L contains an abelian subgroup H which is *decisive* in the sense that the ways in which $\mathcal{T}$ can be weakened and remain Hausdorff are completely determined by the ways in which the relative topology for H can be weakened while remaining Hausdorff and keeping finitely many characters of H continuous. The nature of H depends upon a crucial distinction between (CA) analytic groups (those with closed adjoint image) and non-(CA) analytic groups. Our proof in the latter case employs a homomorphism used by Goto [3] and relies upon structure theorems of Goto [4] and Zerling [18]. The connection between the present paper and certain results which were recently obtained by Goto [5] is discussed in §8. We also note that Hudson's examination of arcwise-connected, finite-dimensional groups [7] leads him to consider, from a different perspective, questions similar to those studied here.

2. Notation and conventions. A topology $\mathcal{U}$ for an abstract group G will be assumed to make the function $f: G \times G \rightarrow G$ given by $f(x, y) = xy^{-1}$ continuous, but $\mathcal{U}$ need not be Hausdorff. For a subgroup H of G , $\mathcal{U}_H$ will denote the relative topology

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induced by $\mathcal{U}$. Since several topologies for the same abstract group G may be under simultaneous consideration, topological statements about G will contain references to the particular topology involved. If $(G, \mathcal{U})$ is a topological group, then $T(G, \mathcal{U})$ will denote the collection of all Hausdorff group topologies for G which are weaker than $\mathcal{U}$. Analytic groups, but not Lie groups, are always connected, and Lie groups do not necessarily satisfy the second axiom of countability.

$\mathbb{Z}$, $\mathbb{R}$, T^q , and $\text{Gl}(V)$ denote, respectively, the integers, the real numbers, the q -dimensional toroid, and the linear automorphism group of a vector space V . T^q , $\text{Gl}(V)$, and finite groups will be assumed to have their usual topologies, unless stated otherwise. The relative topology for a subgroup A of $\text{Gl}(V)$ will be called the *full-linear-group* (*flg*) topology for A . The symbol $\square$ marks the end of a proof.

3. Preliminary definitions and main results. Let $(G, \mathcal{T})$ be a topological group. According to [1, pp. 634–635], the collection of all $\mathcal{T}$ -arc-components of $\mathcal{T}$ -open subsets of G is the basis for a locally arcwise-connected group topology $\mathcal{T}^*$ for G . $(G, \mathcal{T}^*)$ is the *locally arcwise-connected* (*l.a.c.*) group associated with $(G, \mathcal{T})$. For easy reference, we list here some of the properties of l.a.c. groups that are proved in [1].

3.1. THEOREM (GLEASON AND PALAIS [1, 3.2, 4.3, 7.3]). *Let $(G, \mathcal{T})$ be a Hausdorff topological group. Then*

- (i) $\mathcal{T}^*$ is stronger than $\mathcal{T}$, and $(\mathcal{T}^*)^* = \mathcal{T}^*$.
- (ii) $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$ have the same arcs.
- (iii) The $\mathcal{T}^*$ -connected components of a $\mathcal{T}^*$ -open subset X of G are the $\mathcal{T}$ -arc-components of X .
- (iv) If $(G, \mathcal{T})$ is a second-countable Lie group and $\mathcal{U} \in T(G, \mathcal{T})$, then $\mathcal{U}^* = \mathcal{T}$.
- (v) If $(G, \mathcal{T})$ is separable, metrizable, and of finite topological dimension, then $(G, \mathcal{T}^*)$ is a Lie group.

Hudson observes that the assumption of metrizability in (v) may be dropped if by “dimension” one means cohomological dimension [6, p. 68]. We will say that $(G, \mathcal{T})$ is a *weakened Lie* (*WL*) group if $(G, \mathcal{T}^*)$ is a Lie group and $\mathcal{T}$ is Hausdorff. From (iv) we see that the WL groups with which a given analytic group $(L, \mathcal{T})$ is associated are all those of the form $(L, \mathcal{U})$, where $\mathcal{U} \in T(L, \mathcal{T})$.

We now introduce the notion of “decisiveness,” which is central to our main result. Let $(A, \mathcal{A})$ be a topological group with an abstract subgroup B , and let $I(b)$ denote conjugation by an element b of B . (Recall that group topologies are not assumed to be Hausdorff.) If $\mathcal{B}$ is a topology for B that makes the function $f: A \times B \rightarrow A$, defined by $f(a, b) = bab^{-1}$, $(\mathcal{A} \times \mathcal{B}, \mathcal{A})$ -continuous, then $\mathcal{A} \times \mathcal{B}$ is a group topology for the semidirect product $A \times_{\mathcal{I}} B$. We define a homomorphism α of $A \times_{\mathcal{I}} B$ onto A by $\alpha(a, b) = ab$, and the unique topology for A which makes α continuous and open is called the *standard extension* of $\mathcal{B}$ to A , denoted $\mathcal{E}(\mathcal{B})$. We observe that a basis for the neighborhoods of the identity in $\mathcal{E}(\mathcal{B})$ is the collection of all PN , where P and N are, respectively, $\mathcal{A}$ - and $\mathcal{B}$ -neighborhoods of e . $\mathcal{E}(\mathcal{B})$ will be Hausdorff if $\mathcal{A}$ is Hausdorff, B is $\mathcal{A}$ -closed, and $\mathcal{B} \in T(B, \mathcal{A}_B)$.

When $\mathcal{B}$ is weaker than $\mathcal{A}_B$, $\mathcal{E}(\mathcal{B})$ may be thought of as “weakening the topology

of A by weakening that of B'' ; we are interested in situations where this is the *only* way in which $\mathcal{A}$ can be weakened. If B is $\mathcal{A}$ -closed and if $\mathcal{U} = \mathcal{E}(\mathcal{U}_B)$ for every $\mathcal{U} \in T(A, \mathcal{A})$, we will say that B is *decisive* in $(A, \mathcal{A})$. We may now state our main result, the proof of which is contained in §§4–6.

3.2. MAIN THEOREM. *Let $(L, \mathcal{T})$ be a connected Lie group with Lie algebra l .*

(i) *L contains an abelian decisive subgroup H of the form $\mathbb{R}^p \times T^q \times \mathbb{Z}^r \times D$, where p, q , and r are nonnegative integers and D is finite. The adjoint image of H is contained in a toroid Q .*

(ii) *The WL groups with which $(L, \mathcal{T})$ is associated are precisely those of the form $(L, \mathcal{E}(\mathcal{B}))$, where $\mathcal{B} \in T(H, \mathcal{T}_H)$ and the restriction of the adjoint representation, $\text{Ad}: H \rightarrow Q$, is $\mathcal{B}$ -continuous.*

(iii) *If $(L, \mathcal{T})$ is (CA)—that is, if $\text{Ad}(L)$ is a flg-closed subgroup of $\text{Gl}(l)$ —then H is the center of L .*

(iv) *If $(L, \mathcal{T})$ is not (CA), then $H = J \times V$, where V is a vector group, J contains the center $Z(L)$ of L , and $J/Z(L)$ is finite (trivial if L is solvable).*

We conclude this section with an important lemma about standard extensions.

3.3. LEMMA. *Let $(A, \mathcal{A})$ be a topological group with an abstract subgroup B . Let $\mathcal{C}$ be the collection of all topologies $\mathcal{A}'$ for A that are weaker than $\mathcal{A}$ and such that the function $h: A \times A \rightarrow A$ given by $h(a, g) = gag^{-1}$ is $(\mathcal{A} \times \mathcal{A}', \mathcal{A})$ -continuous. If $\mathcal{U} \in \mathcal{C}$ and $\mathcal{U} = \mathcal{E}(\mathcal{U}_B)$, then $\mathcal{A}' = \mathcal{E}(\mathcal{A}'_B)$ for every $\mathcal{A}'$ in $\mathcal{C}$ which is stronger than $\mathcal{U}$.*

PROOF. By hypothesis, the homomorphism $\alpha: A \times_I B \rightarrow A$ given by $\alpha(a, b) = ab$ is $(\mathcal{A} \times \mathcal{U}_B, \mathcal{U})$ -open and continuous. Let $\mathcal{A}' \in \mathcal{C}$ be stronger than $\mathcal{U}$, and let $\{g_i: i \in I\}$ be a net in A which $\mathcal{A}'$ -converges to e . Then $\{g_i\}$ also $\mathcal{U}$ -converges to e and thus has a subnet $\{g_{i(j)}: j \in J\}$ which lifts to a net $\{(a_j, b_j): j \in J\}$ in $A \times_I B$ that is $\mathcal{A} \times \mathcal{U}_B$ -convergent to e . Since $b_j = a_j^{-1}g_{i(j)}$ and $\mathcal{A}'$ is weaker than $\mathcal{A}$, it follows that $b_j \rightarrow e$ in $\mathcal{A}'$, and thus α is $(\mathcal{A} \times \mathcal{A}'_B, \mathcal{A}')$ -open. $\square$

4. Decisive subgroups. The purpose of this section is to prove the first three parts of Main Theorem 3.2; the proof of part (iv) is reserved for §§5 and 6. We begin by proving that every WL group has a continuous adjoint representation.

Let $(G, \mathcal{U})$ be a WL group, and let L be the $\mathcal{U}$ -arc-component of the identity. By 3.1, $(L, (\mathcal{U}^*)_L)$ is an analytic group (whose Lie algebra we denote by l) and $(\mathcal{U}^*)_L = (\mathcal{U}_L)^*$. G acts by inner automorphism on L , and we denote by I and Ad the corresponding homomorphisms of G into $\text{Aut}(L)$ and $\text{Gl}(l)$. If $\text{Aut}(L)$ has the generalized compact-open topology, then I and Ad are $\mathcal{U}^*$ -continuous. We now show that they are also $\mathcal{U}$ -continuous.

4.1. LEMMA. *Let $\lambda: [0, 1] \rightarrow L$ be a $\mathcal{U}_L^*$ -arc with initial point e and let $\{x_j: j \in J\}$ be a net in G which $\mathcal{U}$ -converges to $x \in G$. Then $\{x_j \lambda(t)x_j^{-1}: j \in J\}$ $\mathcal{U}_L^*$ -converges to $x\lambda(t)x^{-1}$, uniformly in t .*

PROOF. Let N be the $\mathcal{U}_L$ -arc-component of e in a $\mathcal{U}_L$ -open neighborhood U of e , and let U_1 be a $\mathcal{U}_L$ -open neighborhood of e such that $U_1^2 \subseteq U$. For each $j \in J$, define

a $\mathcal{U}_L$ -arc $\beta_j: [0, 1] \rightarrow L$ by

$$\beta_j(t) = x\lambda(-t)x^{-1}x_j\lambda(t)x_j^{-1}.$$

For each s in $[0, 1]$, there exist an open neighborhood I_s of s and an element j_s of J such that

$$(1) \quad x\lambda(-s)x^{-1}x_{j_s}\lambda(t)x_{j_s}^{-1} \in U_1$$

and

$$(2) \quad x\lambda(-t)\lambda(s)x^{-1} \in U_1$$

whenever $t \in I_s$ and $j \geq j_s$. From (1) and (2) we see that $\beta_j(t)$ is in U if $t \in I_s$, $j \geq j_s$. By compactness of the unit interval, there is a $j^* \in J$ such that $\beta_j(t) \in U$ for all $t \in [0, 1]$ and for all $j \geq j^*$. For such j , β_j will then be an arc in U that contains the identity, and therefore β_j is in fact an arc in N . Thus $\beta_j(t) \in N$ for all $t \in [0, 1]$ and for all $j \geq j^*$, and the lemma is proved. $\square$

4.2. PROPOSITION. $\text{Ad}: G \rightarrow \text{Gl}(l)$ and $I: G \rightarrow \text{Aut}(L)$ are $\mathcal{U}$ -continuous.

PROOF. Let $\{x_j: j \in J\}$ be a net in G which $\mathcal{U}$ -converges to e , and let W be an open neighborhood of 0 in l such that the exponential mapping $\exp: l \rightarrow L$ is a diffeomorphism on $2W$. To prove that Ad is $\mathcal{U}$ -continuous, it is sufficient to show that $\text{Ad}(x_j)(w) \rightarrow w$ for any $w \in W$. From 4.1 we know that there is an element j^* of J such that, if $j \geq j^*$, then

$$\exp(t \text{Ad}(x_j)(w)) = x_j(\exp tw) x_j^{-1} \in \exp W$$

for all $t \in [0, 1]$, and it follows that $\text{Ad}(x_j)(w) \in W$ if $j \geq j^*$. Lemma 4.1 also implies that, in $\mathcal{U}_L^*$,

$$\exp(\text{Ad}(x_j)(w)) = x_j(\exp w) x_j^{-1} \rightarrow \exp w.$$

Since $\exp$ is a diffeomorphism on W , we conclude that $\text{Ad}(x_j)(w) \rightarrow w$ in l , as desired.

To prove that I is $\mathcal{U}$ -continuous, we simply observe that the differential operator d is a topological group isomorphism of $\text{Aut}(L)$ onto a subgroup of $\text{Gl}(l)$. This completes the proof of 4.2. $\square$

We now deduce an important criterion by which decisive subgroups of analytic groups may be identified.

4.3. LEMMA. Let $(L, \mathcal{T})$ be an analytic group, B a $\mathcal{T}$ -closed subgroup of L , and $\mathcal{U}$ the weakest (not necessarily Hausdorff) topology for L that makes Ad $\mathcal{U}$ -continuous. If $\mathcal{U} = \mathcal{E}(\mathcal{U}_B)$, then B is decisive in L .

PROOF. According to 3.1 and 4.2, every topology in $T(L, \mathcal{T})$ is stronger than $\mathcal{U}$. We may then apply 3.3, with $A = L$ and $\mathcal{A} = \mathcal{T}$. $\square$

If $(L, \mathcal{T})$ is a (CA)analytic group, then $\text{Ad}(L)$ is closed in $\text{Gl}(l)$. If $(L, \mathcal{T})$ is not (CA), then $\text{Gl}(l)$ contains a toral subgroup Q such that the flg-closure C of $\text{Ad}(L)$ equals $\text{Ad}(L) \cdot Q$. (See, for example, Goto [2, Theorem 1].) Thus in either case, $C = \text{Ad}(L) \cdot Q$, where Q is a (possibly trivial) toral subgroup of $\text{Gl}(l)$.

4.4. PROPOSITION. *Let $(L, \mathcal{T})$ be an analytic group with Lie algebra l , let C be the flg-closure of $\text{Ad}(L)$, and let Q be a (possibly trivial) toral subgroup of $\text{Gl}(l)$ such that $C = \text{Ad}(L) \cdot Q$. Then $\text{Ad}^{-1}(Q)$ is an abelian decisive subgroup of L and has the form $\mathbf{R}^p \times T^q \times \mathbf{Z}^r \times D$, where p, q , and r are nonnegative integers and D is finite.*

PROOF. Let $H = \text{Ad}^{-1}(Q)$. Because $d: \text{Aut}(L) \rightarrow d(\text{Aut}(L))$ is a topological isomorphism whose image is flg-closed and includes $\text{Ad}(L)$, there is a toral subgroup K of $\text{Aut}(L)$ such that $Q = dK$, and thus $H = I^{-1}(K)$. According to Goto [3, Lemma 4], each automorphism in K leaves H pointwise fixed. In particular, since $I(H) \subseteq K$, H is abelian and therefore has the form $\mathbf{R}^p \times T^q \times E$, where p and q are nonnegative integers and E is discrete. We now invoke Theorem 1' and the subsequent remark in Mostow [8] to show that E is finitely generated and thus equals $\mathbf{Z}^r \times D$, where r is a nonnegative integer and D is finite.

It remains to show that H is decisive in L . If $\mathcal{U}$ is the weakest topology for L which makes $\text{Ad}(\mathcal{U}, \text{flg})$ -continuous, then it suffices, by 4.3, to show that $\mathcal{U} = \mathcal{E}(\mathcal{U}_H)$. To do so, we form the semidirect product $L \ltimes K$ and define a homomorphism $\psi: L \ltimes K \rightarrow C$ by $\psi(x, k) = \text{Ad}(x) \cdot dk$ for $x \in L, k \in K$. If $\mathcal{W}$ is the topology which K inherits from the generalized compact-open topology, then ψ is surjective, $(\mathcal{T} \times \mathcal{W}, \text{flg})$ -continuous, and thus $(\mathcal{T} \times \mathcal{W}, \text{flg})$ -open onto its image C . It follows that a basis for the neighborhoods of e in the flg-topology for C is the collection of all $\text{Ad}(P) \cdot dN$, where P is a $\mathcal{T}$ -neighborhood of e in L and N is a $\mathcal{W}$ -neighborhood of e in K , and therefore a basis for the $\mathcal{U}$ -neighborhoods of e is the collection of all $\text{Ad}^{-1}(\text{Ad}(P) \cdot dN)$. Now for each such P and N , the fact that Ad is a homomorphism implies that

$$\text{Ad}^{-1}(\text{Ad}(P) \cdot dN) = P \cdot \text{Ad}^{-1}(dN).$$

Since $\text{Ad}^{-1}(dN)$ is simply a $\mathcal{U}_H$ -neighborhood of e , we have shown that $\mathcal{U} = \mathcal{E}(\mathcal{U}_H)$. $\square$

Proposition 4.4 proves part (i) of the Main Theorem, and we may now proceed to prove parts (ii) and (iii). Let H be any abelian decisive subgroup of $(L, \mathcal{T})$. We will call a topology $\mathcal{B}$ for H *allowable* if $\mathcal{B} \in T(H, \mathcal{T}_H)$ and the restriction $\text{Ad}: H \rightarrow \text{Gl}(l)$ is $\mathcal{B}$ -continuous. If $(L, \mathcal{U})$ is a Hausdorff topological group and $\mathcal{U}^* = \mathcal{T}$, then $\mathcal{U} = \mathcal{E}(\mathcal{U}_H)$, and by 4.2 $\mathcal{U}_H$ is an allowable topology for H . On the other hand, if $\mathcal{B}$ is such a topology then $\mathcal{E}(\mathcal{B})$ is a group topology for L which is in $T(L, \mathcal{T})$ and by 3.1(iv) $(L, \mathcal{T})$ is the l.a.c. group associated with $(L, \mathcal{E}(\mathcal{B}))$. To prove part (iii) of 3.2, we simply note that when $(L, \mathcal{T})$ is (CA) we may choose the trivial toroid for Q , so that the decisive subgroup $\text{Ad}^{-1}(Q)$ will be the center of L .

5. Subgroups of $\text{Gl}(n, \mathbf{R})$. Before proving part (iv) of the Main Theorem, which is the purpose of §6, we must examine in detail the structure of the flg-closure of $\text{Ad}(L)$ when $(L, \mathcal{T})$ is not (CA). The basis for our discussion is a result in Goto [4], which we now describe. Let $(G, \mathcal{U})$ be an analytic subgroup of $\text{Gl}(n, \mathbf{R})$ which is not flg-closed, with C denoting the flg-closure of G . Let N be any subgroup of G which is maximal among those that are $\mathcal{U}$ -connected, flg-closed, and contain the commutator subgroup D of G . (Such groups exist because, by Lemma 7 in [2], D is flg-closed.) If T_1 is the radical of a maximal flg-compact subgroup of C , let T_2 be the flg-connected component of the identity in $N \cap T_1$ and T a toroid such that $T_1 = T_2 \cdot T$, $T_2 \cap T = \{e\}$.

In our notation, Goto's theorem may be stated as follows:

5.1. THEOREM (GOTO [4, p. 197]). $N \cap T$ is finite, $C = N \cdot T$, and G contains a $\mathcal{U}$ -closed vector subgroup W , the flag-closure of which is T , such that $G = N \cdot W$, $N \cap W = \{e\}$. C is $(\text{flag}, \text{flag} \times \mathcal{U}_N)$ -diffeomorphic with $T \times N$.

Although the groups N and T are not in general unique, we will prove the following.

5.2. PROPOSITION. If N and T are chosen in the manner described, then

- (i) the dimensions of N , W , and T are uniquely determined;
- (ii) the finite groups $N \cap T$ are all isomorphic;
- (iii) $N \cap T$ is trivial if G is solvable.

The proposition depends upon a lemma about abelian analytic groups, the straightforward proof of which we omit.

5.3. LEMMA. Let $(A, \mathcal{A})$ be an abelian analytic group with a dense analytic subgroup $(B, \mathcal{B})$. Let C be a maximal $\mathcal{B}$ -connected and $\mathcal{A}$ -closed subgroup of B . Then the dimension of C equals the sum of the dimensions of the vector part of $(A, \mathcal{A})$ and the compact part of $(B, \mathcal{B})$, and the intersection of C with the compact part of $(A, \mathcal{A})$ is independent of the choice of C .

PROOF OF 5.2. Let $\mathcal{U}'$ and $\mathcal{F}'$ denote the quotient topologies for G/D and C/D obtained from $\mathcal{U}$ and the flag-topology. To prove (i) it will suffice to show that N/D is maximal among the subgroups of G/D which are $\mathcal{U}'$ -connected and $\mathcal{F}'$ -closed, for then 5.3 will assure that the dimension of N/D , and thus of N , is uniquely determined (N/D must be $\mathcal{F}'$ -closed because it is the inverse image of the identity under the projection $C/D \rightarrow C/N$, which is continuous if each group is given the quotient topology from the flag-topology.) That N/D is, in fact, maximal follows from the maximality of N and the $\mathcal{U}$ -connectedness of D .

To prove (ii), we first show that the intersection $N \cap T_1$ is, for given T_1 , independent of the choice of N . For if N_1 and N_2 are two choices for N , then N_1/D and N_2/D are, as we have seen, maximal among the $\mathcal{U}'$ -connected subgroups of G/D which are $\mathcal{F}'$ -closed, and so 5.3 implies that the intersections of N_1/D and N_2/D with the image of T_1 in C/D are equal. Therefore $N_1 \cap DT_1 = N_2 \cap DT_1$, whence $N_1 \cap T_1 = N_2 \cap T_1$.

Since T_1 , as the radical of a maximal flag-compact subgroup of C , is determined up to conjugation by an element of C , and since N is normal in C , the groups $N \cap T_1$ are thus all flag-isomorphic, regardless of the choices of N and T_1 . We complete the proof of (ii) by showing that, for fixed N and T_1 , the isomorphism class of the finite group $N \cap T$ is independent of the choice of T . This follows from the fact that, for any choice of T , $N \cap T_1 = T_2(N \cap T)$ and $T_2 \cap (N \cap T) = \{e\}$.

Finally, we prove (iii). It clearly suffices to show that $N \cap T_1$ is flag-connected when G is solvable. Since G and C have the same commutator subgroup D , C also is solvable. Applying Proposition 2.4 of Van Est [17], we find that $G = B \cdot A$, where B is a flag-closed and simply-connected group containing D , the flag-closure F of A is a toroid, $C = B \cdot F$, and $B \cap F = \{e\}$. We may choose N and a maximal flag-compact subgroup

K of C in such a way that $B \subseteq N$, $F \subseteq K$. That the latter inclusion is in fact an equality follows from the easily verified equation $K = (B \cap K) \cdot F$ and the fact that B , as a simply-connected and solvable Lie group in the flg-topology, contains no nontrivial compact subgroups. We may therefore let T_1 equal F , and the proof is completed by observing that $N \cap F$ must be flg-connected, since $N = B(N \cap F)$, N and B are flg-connected, and $B \cap (N \cap F) = B \cap F$ is trivial. $\square$

Applying 5.1 and 5.2 to the adjoint image of a non-(CA) analytic group, we obtain the following.

5.4. COROLLARY. *Let $(L, \mathcal{T})$ be a non-(CA) analytic group, $\mathcal{U}$ the Lie topology for $\text{Ad}(L)$, and C the flg-closure of $\text{Ad}(L)$. If N is a maximal $\mathcal{U}$ -connected and flg-closed subgroup of $\text{Ad}(L)$ containing the commutator subgroup of $\text{Ad}(L)$, T_1 is the radical of a maximal flg-compact subgroup of C , and T is a toroid in T_1 complementary to the flg-connected component of the identity in $N \cap T_1$, then $\text{Ad}(L)$ contains a $\mathcal{U}$ -closed vector subgroup W , whose flg-closure is T , such that $\text{Ad}(L) = N \cdot W$, $C = N \cdot T$, and $N \cap W$ is trivial. Moreover, $N \cap T$ is a finite group whose isomorphism class does not depend on the particular choices of N and T . $N \cap T$ is trivial if $\text{Ad}(L)$ is solvable, and the dimensions of N , W , and T are independent of the choices of N and T .*

6. The non-(CA) case. The proof of part (iv) of the Main Theorem, which is contained in this section, relies not only upon 5.4 but also upon Zerling's structure theorem for non-(CA) analytic groups [18], which says that every such group is the semidirect product of a (CA) analytic group and a vector group. More precisely, the relevant portion of Zerling's results may be stated as follows.

6.1. THEOREM (ZERLING [18, THEOREM 2.1]). *With notation as in 5.4, let P be the $\mathcal{T}$ -connected component of the identity in $\text{Ad}^{-1}(N)$. Then $(P, \mathcal{T}_P)$ is (CA) and contains the center of L , and $\text{Aut}(P)$ contains a vector subgroup V with compact closure such that, if $\mathcal{V}$ is the vector topology for V , then $P \circledcirc V$ is $(\mathcal{T}_P \times \mathcal{V}, \mathcal{T})$ -isomorphic with L . If we identify L with $P \circledcirc V$, then $\text{Ad}(V) = W$, and V and W have the same dimension.*

Adopting the notation of 5.4 and 6.1, we now combine these results with 4.4 to prove part (iv) of the Main Theorem. Since $C = \text{Ad}(L) \cdot T$ by 5.4, it follows from 4.4 that $H = \text{Ad}^{-1}(T)$ is an abelian decisive subgroup of L . If we let J equal $P \cap \text{Ad}^{-1}(N \cap T)$, a trivial computation verifies that $H = J \circledcirc V$. Since H is abelian, V must act trivially on J , whence $H = J \times V$. To complete the proof, we note that $\text{Ad}(P)$ must equal all of N , since $N \cap \text{Ad}(V) = N \cap W$ is trivial. Therefore $\text{Ad}(J) = N \cap T$ and $J/Z(L)$ is isomorphic to the finite group $N \cap T$. If L is solvable, then so is $\text{Ad}(L)$, and 5.4 assures that $J/Z(L)$ is trivial. This completes the proof of the Main Theorem. We may also note that, according to 5.4 and 6.1, the dimension of V and the isomorphism class of $J/Z(L)$ are independent of the particular choices of N and T .

7. Examples. We now give an example of a non-(CA) analytic group whose center, in the notation of the Main Theorem, has index two in the group J . Define an action of $SU(2) \times T^2$ on $\mathbb{C}^3$ by letting $(A, e^{i\theta}, e^{i\psi})$, where $A \in SU(2)$, correspond to the 3×3 complex matrix in Figure 1. The semidirect product $G = \mathbb{C}^3 \circledcirc (SU(2) \times T^2)$

is then a Lie group in its usual topology. If μ is some fixed irrational number, then $L = \mathbb{C}^3 \oplus (SU(2) \times \mathbb{R})$ becomes a dense Lie subgroup of G by means of the injection $\alpha: (z, A, t) \mapsto (z, A, e^{it}, e^{i\mu t})$, $z \in \mathbb{C}^3$, $A \in SU(2)$, $t \in \mathbb{R}$. L has trivial center and thus cannot, by Theorem 2.2.1 of Van Est [16], be (CA).

By considering the action of G on L by inner automorphism, one verifies that $G/Z(G)$ is, in the quotient topology, isomorphic with the flag-closure of $\text{Ad}(L)$, so that we may regard the adjoint representation of L as the composition of α with the projection $\pi: G \rightarrow G/Z(G)$. Letting $N = \pi(\mathbb{C}^3 \times SU(2) \times 1 \times 1)$ and $T_1 = \pi(0 \times I \times T^2)$, we find that $N \cap T_1$ is a two-element group. Therefore $T = T_1$ and $\text{Ad}^{-1}(T) = 0 \times J \times \mathbb{R}$, where J is the group generated by $-I$.

$$\begin{array}{c|c} & 0 \\ & 0 \\ \hline 0 & 0 \end{array} \quad e^{i\psi}$$

FIGURE 1

We also observe that the reduction to the abelian case which the Main Theorem effects does not prevent WL groups from having rather peculiar topologies. Although the author will undertake a systematic study of “unusual” topologies for abelian groups in a subsequent paper [15], we may note here that any sequence in $\mathbb{R}^n$ or $\mathbb{Z}^n$ which “goes to infinity sufficiently fast” will, in an appropriately weakened topology, converge to 0. For example,

$$d(n, m) = \inf \{ \sum |c_i|/i! |n - m = \sum c_i(i! + 1), c_i \in \mathbb{Z} \}$$

defines a metric on $\mathbb{Z}$ in which $i! + 1 \rightarrow 0$. Other examples of unusual topologies for $\mathbb{Z}$ and $\mathbb{R}^n$ can be found in [9–14].

8. Related results. After writing this paper, the author learned of related but independent results obtained by Goto [5], and in this section we will sketch the connection between his work and ours. Let $(L, \mathcal{T})$ and $(G, \mathcal{U})$ be, respectively, an analytic and a topological group, and let $f: L \rightarrow G$ be a continuous, injective homomorphism. Although the course of Goto’s analysis parallels our own in certain respects, his primary interest is a description of the set $\overline{f(L)}$, while our Main Theorem can be viewed as a characterization of the topology of $f(L)$. Changing Goto’s notation to distinguish the *gm*-torus in [5] from our own decisive subgroup H , we can summarize his principal results as follows: If $v(J)$ is the vector part of the *gm*-torus J , then $\overline{f(L)} = f(L) \overline{f(v(J))}$, and f is an imbedding (i.e., a homeomorphism of L onto $f(L)$) if and only if $f|v(J)$ is. We note that the equation $\overline{f(L)} = f(L) \overline{f(v(J))}$ is a statement about sets, not topologies, and that it is trivially valid when $f(L) = G$.

Now our decisive subgroup H , besides revealing whether f is an imbedding, also

explicitly determines the topology of $f(L)$ when f is not an imbedding; identifying L with $f(L)$, we know that $\mathcal{U}_L = \mathcal{E}(\mathcal{U}_H)$. The following example illustrates the relationship among H , J , and $\nu(J)$. If $L = \mathbf{R} \times T^1 \times SU(2)$, then $H = \mathbf{R} \times T^1 \times \{\pm I\}$, $J = \mathbf{R} \times T^1 \times O(2)$, and $\nu(J) = \mathbf{R} \times 1 \times I$. As the author will show in [15], there is a topology $\mathcal{U}$ for L , weaker than the usual topology, in which $\{(n!, (-1)^n, I)\}$ converges to the identity. Clearly $\mathcal{U}$ cannot equal $\mathcal{E}(\mathcal{U}_{\nu(J)})$, and thus $\nu(J)$ is "too small" to determine the topology of L ; indeed, a theorem in [15] shows that in this case H is the smallest subgroup that will suffice. On the other hand, J is "too big", because it properly contains H .

Finally, we note that Theorem 1 in [5] provides a somewhat more elegant proof of our 4.2 and that Theorem 2 in [5] is a sharper form of the theorem in [4] which we cited in §5. The latter makes possible the following improvement in our Main Theorem: When L is not (CA), we can alter the choice of Q to assure that $H = Z(L) \times V$, even if L is not solvable. The details are contained in [15].

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